DISCRETE MATHEMATICS 10 (1974) 75–92. © North-Holland Publishing Company

COHOMOLOGY OF INCIDENCE COALGEBRAS

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Received 4 March 1974

Abstract. The cohomology groups $H^n(M, C)$ are studied, where C is the incidence coalgebra of a locally finite partially ordered set P and where M is a C-C comodule depending on a convex coarsening of the given partial order on P. The case where P is a geometric lattice and the convex coarsening is just the equality relation is emphasized.

1. The incidence coalgebra and some associated two-sided comodules

Let $C = C(P)$ be the incidence coalgebra, over any field K of characteristic zero, of the locally finite partially ordered set P. That is, C is the K-vector space with basis

$$\{(x, y) : x, y \in P, x \leqslant y\}$$

and with coalgebra structure maps

$$\Delta : C \to C \otimes C, \qquad \epsilon : C \to K$$

determined by

$$\Delta(x, y) = \sum_{x \leqslant t \leqslant y} (x, t) \otimes (t, y),$$

$$\epsilon(x, y) = \delta_{xy} \qquad \text{(Kronecker's delta)},$$

where $x \leqslant y$ in P.

We introduce a generalization due to David Smith of the C-C comodules studied in [2]. A partial order relation σ on P is said to be a convex coarsening of the (fixed) partial order relation $\leqslant$ on P provided

$$x \,\sigma\, y \Rightarrow x \leqslant y,$$

$$x \,\sigma\, y,\ x \leqslant t \leqslant y \Rightarrow x \,\sigma\, t \,\sigma\, y.$$

On the vector space C, the associations

$$(x, y) \mapsto \sum_{x\,\sigma\, t \leqslant y} (x, t) \otimes (t, y), \qquad (x, y) \mapsto \sum_{x \leqslant t\, \sigma\, y} (x, t) \otimes (t, y)\,,$$

where $x \leqslant y$ in P, determine linear maps λ and ρ respectively, which give C the structures of left C-comodule and right C-comodule. Moreover,

$$(I \otimes \rho) \circ \lambda = (\lambda \otimes I) \circ \rho\,,$$

so that the left and right structures are compatible. If we write C_σ rather than C, then C_σ is a C-C comodule (viz. compatibly a left and right C-comodule) with structure maps

$$\lambda : C_\sigma \to C \otimes C_\sigma, \qquad \rho : C_\sigma \to C_\sigma \otimes C\,.$$

The convex coarsenings of $\leqslant$ form a complete lattice. The bottom element of this lattice is the relation of equality, which determines the C-C comodule $C_=$ with left and right structure maps determined by

$$(x, y) \mapsto (x, x) \otimes (x, y), \qquad (x, y) \mapsto (x, y) \otimes (y, y)\,,$$

respectively. The top element of the lattice is of course $\leqslant$. As C-C comodules, $C_\leqslant = C$ so we may supress the subscript in this case.

Another example of a convex coarsening of $\leqslant$ arises from a partition σ of P into convex subsets. In this case we abuse notation and write $x\ \sigma\ y$ to mean that $x \leqslant y$ and that x and y are in the same block of σ.

As a final example of a convex coarsening of $\leqslant$, consider an increasing map $\sigma : P \to Q$ of partially ordered sets and again abuse notation to let $x\ \sigma\ y$ mean that $x \leqslant y$ and $\sigma(x) = \sigma(y)$.

2. Cohomology of the incidence coalgebra

We will study the cohomology groups $H^n(M, C)$ for selected C-C comodules M. This cohomology is dual to the Hochschild cohomology of the incidence algebra. We work with the incidence coalgebra rather than its dual algebra because the coalgebra has the intrinsic basis of all (x, y), $x \leqslant y$ in P, facilitating many computations. Our results may be dualized.

We describe this cohomology in terms of K-linear maps $f : M \to C \otimes \ldots \otimes C$. Define

$$\partial_n^{\,i} = I^{i-1} \otimes \Delta \otimes I^{n-i} : C^n \to C^{n+1}, \qquad 1 \leqslant i \leqslant n,$$

$$\epsilon_n^{\,i} = I^{i-1} \otimes \epsilon \otimes I^{n-i} : C^n \to C^{n-1}, \qquad 1 \leqslant i \leqslant n,$$

where C^n denotes the n-fold tensor product of C and $C^0 = K$. For any C-C comodule M with left and right structure maps λ and ρ respectively, define

$$\delta_M : \mathrm{Hom}_K(M, C^n) \to \mathrm{Hom}_K(M, C^{n+1})$$

for each $n = 0, 1, 2, \ldots$, by

$$\delta_M(g) = (I \otimes g) \circ \lambda + \sum_{i=1}^{n} (-1)^i \partial_n^{\,i} \circ g + (-1)^{n+1} (g \otimes I) \circ \rho .$$

Let

$$N^n(M, C) = \{g \in \mathrm{Hom}_K(M, C^n) \colon \epsilon_n^{\,i}(g) = 0,\ i = 1, \ldots, n\}$$

for $n \geqslant 1$ and $N^0(M, C) = \mathrm{Hom}_K(M, K)$. Then δ_M induces a K-linear map

$$\delta_M : N^n(M, C) \to N^{n+1}(M, C),$$

and we let

$$Z^n(M, C) = \mathrm{Ker}(\delta_M : N^n(M, C) \to N^{n+1}(M, C)),$$

$$B^n(M, C) = \mathrm{Im}(\delta_M : N^{n-1}(M, C) \to N^n(M, C)),$$

with the understanding that $B^0(M, C) = (0)$. Then $H^0(M, C) = Z^0(M, C)$, and for $n \geqslant 1$,

$$H^n(M, C) = Z^n(M, C)/B^n(M, C).$$

For complete details of this cohomology, the reader is referred to [3]. To minimize notation, we will write δ rather than δ_M.

3. Computation of $H^0(C_\sigma, C)$

Let σ be a convex coarsening of $\leqslant$ on P, and let $g \in N^0(C_\sigma, C)$. Then

$$(3.1) \qquad \delta g(x, y) = \sum_{x \sigma u \leqslant y} g(u, y)\,(x, u) - \sum_{x \leqslant t \sigma y} g(x, t)\,(t, y) .$$

Theorem 3.1. dim $H^0(C_\sigma, C)$ = *cardinality of the set of connected components of the partial order* σ.

Proof. Let $g \in Z^0(C_\sigma, C)$. So $\delta g(x, y) = 0$ for $x \leqslant y$. If $x < y$, then $g(x, y)$ is the coefficient of (x, x) in (3.1), and hence $g(x, y) = 0$. If $x \ \sigma \ y$, then $g(x, x) = g(y, y)$ since $g(y, y) - g(x, x)$ is the coefficient of (x, y) in (3.1). So $g(y, y) = g(x, x)$ when x and y are in the same connected component of σ, and the theorem follows.

Corollary 3.2. dim $H^0(C, C)$ = *cardinality of the set of connected components of the partial order* $\leqslant$ *on* P.

Corollary 3.3. dim $H^0(C_=, C)$ = *cardinality of* P.

Corollary 3.4. *If* σ *is a partition of* P *into convex subsets (with respect to* $\leqslant$*)* $\sigma_1, \sigma_2, \ldots, \sigma_n$, *then*

$$\dim H^0(C_\sigma, C) = \sum_{i=1}^{n} (\text{cardinality of the set of connected components of } \sigma_i).$$

Corollary 3.5. *If* $P \xrightarrow{\sigma} Q \xrightarrow{\tau} P$ *is a residuated pair of increasing maps determining the closure operator* $J = \tau \circ \sigma$ *on* P, *then* $C_J = C_\sigma$ *and*

$$\dim H^0(C_J, C) = \text{cardinality of the set of } J\text{-closed elements of } P.$$

Proof. That $C_J = C_\sigma$ follows from the fact that $\sigma = \sigma \circ J$.

4. Computations for $H^1(C_\sigma, C)$

Let $f \in N^1(C_\sigma, C)$, and write, for $x \leqslant y$,

$$f(x, y) = \sum_{u \leqslant v} \alpha^{xy}{}_{uv}(u, v)\,.$$

Then

$$\sum_{u} \alpha^{xy}{}_{uu} = 0 \quad \text{for any } x \leqslant y\text{ , and} \tag{4.1}$$

$$(4.2) \qquad \delta f(x,y) = \sum_{\substack{x\,\sigma\, r \leq y \\ z \leq w}} \alpha^{ry}{}_{zw}(x,r)\otimes(z,w) - \sum_{u\leq t\leq v} \alpha^{xy}{}_{uv}(u,t)\otimes(t,v)$$

$$+ \sum_{\substack{x \leq s\, \sigma\, y \\ p \leq q}} \alpha^{xs}{}_{pq}(p,q)\otimes(s,y) \qquad \text{for } x \leq y\,.$$

Assume that $f \in Z^1(C_\sigma, C)$ so that $\delta f(x,y) = 0$. Inspecting the coefficient of $(u,u)\otimes(u,v)$ in (4.2), we see that

$$(4.3) \qquad \alpha^{xy}{}_{uv} = 0 \quad \text{when both } u \neq x \text{ and } v \neq y\,.$$

From (4.1) and (4.3) we conclude that

$$(4.4) \qquad \alpha^{xy}{}_{xx} = -\alpha^{xy}{}_{yy} \quad \text{and} \quad \alpha^{xx}{}_{xx} = 0\,.$$

Continuing to use the fact that the expression in (4.2) is 0, we get

$$(4.5) \qquad \alpha^{xy}{}_{xv} = 0 \quad \text{for } v \nleq y$$

(see coefficient of $(x,v)\otimes(v,v)$),

$$(4.6) \qquad \alpha^{xy}{}_{uy} = 0 \quad \text{for } x \nleq u$$

(see coefficient of $(u,u)\otimes(u,y)$),

$$(4.7) \qquad \alpha^{xy}{}_{xv} = \begin{cases} \alpha^{ty}{}_{tv} & \text{for } x\ \sigma\ t\ \sigma\ v < y, \\ 0 & \text{for } x \not\sigma\ v < y \end{cases}$$

(see coefficients of $(x,t)\otimes(t,v)$ and $(x,v)\otimes(v,v)$),

$$(4.8) \qquad \alpha^{xy}{}_{uy} = \begin{cases} \alpha^{xt}{}_{ut} & \text{for } x < u\ \sigma\ t\ \sigma\ y\,, \\ 0 & \text{for } x < u \not\sigma\ y \end{cases}$$

(see coefficients of $(u,t)\otimes(t,y)$ and $(u,u)\otimes(u,y)$),

$$(4.9) \qquad \alpha^{xy}{}_{xy} = \begin{cases} \alpha^{xt}{}_{xt} + \alpha^{ty}{}_{ty} & \text{for } x\ \sigma\ t\ \sigma\ y, \\ \alpha^{xt}{}_{xt} & \text{for } x \not\sigma\ t\ \sigma\ y, \\ \alpha^{ty}{}_{ty} & \text{for } x\ \sigma\ t \not\sigma\ y, \\ 0 & \text{if there exists } t \text{ with } x < t < y,\ x \not\sigma\ t \not\sigma\ y \end{cases}$$

(see coefficient of $(x,t)\otimes(t,y)$ in each case).

Lemma 4.1. *For any $f \in Z^1(C_\sigma, C)$ as above, there exists $g \in N^0(C_\sigma, C)$*

such that

$$(f - \delta g)(x, y) = \alpha^{xy}{}_{xy}(x, y)$$

for $x \leqslant y$.

Proof. For $x \leqslant y$, define $g(x, y) = \alpha^{xy}{}_{xx}$ and notice that $g(x, x) = 0$ follows from (4.4). So in view of (4.3), (4.5) and (4.6), we need only show that the coefficients in $(f - \delta g)(x, y)$ of (x, v) and (u, y), where $v < y$ and $x < u$, are 0. But the coefficients of such an (x, v) is computed from (3.1) and the definition of g to be

$$\alpha^{xy}{}_{xv} - \alpha^{vy}{}_{vv} = 0 \qquad \text{for } x \; \sigma \; v \text{ (see (4.7))},$$

and to be 0 for $x \not\sigma\, v$ (see (4.7)). The coefficient of such a (u, y) is 0 if $u \not\sigma\, y$ (see (4.8)) and if $u \; \sigma \; y$ is

$$\begin{aligned} \alpha^{xy}{}_{uy} + \alpha^{xu}{}_{xx} &= \alpha^{xy}{}_{uy} - \alpha^{xu}{}_{uu} && \text{(see (4.4))} \\ &= 0 && \text{(see (4.8))}. \end{aligned}$$

Lemma 4.2. *Suppose each connected component of* σ *has either a top element or a bottom element. Then for any* $f \in Z^1(C_\sigma, C)$, *there exists* $g \in N^0(C_\sigma, C)$ *such that*

$$(f - \delta g)(x, y) = (1 - \zeta\sigma(x, y))\, \alpha^{xy}{}_{xy}(x, y).$$

Here $\zeta\sigma$ *is the zeta function for the partial order* σ.

Proof. By Lemma 4.1, we may assume that $f(x, y) = \alpha^{xy}{}_{xy}(x, y)$ for $x \leqslant y$. For $x < y$, set $g(x, y) = 0$. Suppose for $x \in P$ that the connected component to which x belongs, relative to σ, has a bottom element z. Then set $g(x, x) = \alpha^{zx}{}_{zx}$. For any $y \geqslant x$, we then have from (3.1) that

$$\begin{aligned} (f - \delta g)(x, y) &= [\alpha^{xy}{}_{xy} - \zeta\sigma(x, y)(g(y, y) - g(x, x))](x, y) \\ &= [\alpha^{xy}{}_{xy} - \zeta\sigma(x, y)(\alpha^{zy}{}_{zy} - \alpha^{zx}{}_{zx})](x, y) \\ &= [\alpha^{xy}{}_{xy} - \zeta\sigma(x, y)\, \alpha^{xy}{}_{xy}](x, y), \end{aligned}$$

the last equality following from (4.9).

If, on the other hand, the connected component to which x belongs, relative to σ, has a top element u, then we must set $g(x, x) = -\alpha^{xu}{}_{xu}$. The result again follows from (4.9).

Theorem 4.3. *If every connected component of the partial order $\leqslant$ on P has either a top or a bottom element, then $H^1(C, C) = 0$.*

Proof. This follows immediately from Lemma 4.2 taking σ to be $\leqslant$.

We state in more familiar, but equivalent, terms, the dual version of this theorem as a corollary.

Corollary 4.4. *If P is a locally finite partially ordered set each of whose connected components has either a top or a bottom element and if A is the incidence algebra of P, then every derivation of A in A is inner.*

We say that $x \leqslant y$ is a covering pair provided $x \leqslant t \leqslant y$ implies either $x = t$ or $t = y$.

Theorem 4.5. $\dim H^1(C_=, C)$ = *cardinality of the set of all covering pairs relative to* $\leqslant$.

Proof. This follows immediately from Lemma 4.2, (3.1) and (4.9).

Theorem 4.6. *If $P \xrightarrow{\sigma} Q \xrightarrow{\tau} P$ is a residuated pair of increasing maps determining the closure operator $J = \tau \circ \sigma$ on P, then*

$$\dim H^1(C_J, C) = \text{cardinality of the set of all covering pairs among the } J\text{-closed elements of } P.$$

Proof. Let $f \in Z^1(C_J, C)$. By Lemma 4.2, we may assume that $f(x, y) = 0$ for $x \leqslant y$ and $J(x) = J(y)$ and that $f(x, y) = \alpha^{xy}{}_{xy}(x, y)$ for $x \leqslant y$ and $J(x) \neq J(y)$. In the latter case, we apply (4.9) to get

$$\alpha^{xy}{}_{xy} = \alpha^{xJ(y)}{}_{xJ(y)} = \alpha^{J(x)J(y)}{}_{J(x)J(y)} \, .$$

We also conclude from (4.9) that the above coefficient is 0 unless $(J(x), J(y))$ is a covering pair. But if $x \leqslant y$ and $J(x) \neq J(y)$, the coefficient of (x, y) in $\delta g(x, y)$ is 0 for any $g \in N^0(C_J, C)$ (see (3.1)). The theorem follows.

5. Computations for $H^k(C_=, C)$

Let $M(x, y)$ denote the one-dimensional subspace of C generated by

(x, y), where $x \leqslant y$ in P. It is clear that $M(x, y)$ is a C-C subcomodule of $C_=$ and that

$$C_= = \bigoplus_{x \leqslant y} M(x, y)$$

as C-C comodule. Hence, to compute $H^k(C_=, C)$ it suffices to compute $H^k(M(x, y), C)$ for all $x \leqslant y$ in P.

For the case $k = 0$, the proof of Theorem 3.1, which gave Corollary 3.3, is easily adapted to the case at hand to yield:

Theorem 5.1. $\dim H^0(M(x, y), C) = \delta_{x,y}$ *(Kronecker's delta).*

The proofs of Lemmas 4.1 and 4.2 are easily modified to the present situation for $k = 1$ to give, in analogy with Theorem 4.5, the following result.

Theorem 5.2. $\dim H^1(M(x, y), C) = 1$ *if y covers x and is* 0 *otherwise.*

Now assume that $k \geqslant 2$. Let $f \in N^k(M(x, y), C)$ and write

$$f(x, y) = \sum \alpha_{(u_1, v_1)(u_2, v_2)\ldots(u_k, v_k)}(u_1, v_1) \otimes (u_2, v_2) \otimes \ldots \otimes (u_k, v_k),$$

where the sum is over all k-fold tensors of basis elements of C (e.g. $u_i \leqslant v_i$ in P for $i = 1, 2, \ldots, k$) and all but finitely many of the coefficients are 0. Since $\epsilon_k{}^i(f) = 0$ for $i = 1, 2, \ldots, k$,

$$\sum_{z \in P} \alpha_{(u_1, v_1)\ldots(u_{i-1}, v_{i-1})(z, z)(u_{i+1}, v_{i+1})\ldots(u_k, v_k)} = 0 \tag{5.1}$$

for each $i = 1, 2, \ldots, k$ and each choice of $k - 1$ basis elements $(u_1, v_1), \ldots, (u_{i-1}, v_{i-1}), (u_{i+1}, v_{i+1}), \ldots, (u_k, v_k)$ of C. The definition of δ gives

(5.2)

$$\begin{aligned}\delta f(x, y) = &\sum \alpha_{(u_1, v_1)\ldots(u_k, v_k)}(x, x) \otimes (u_1, v_1) \otimes \ldots \otimes (u_k, v_k)\\ &+ \sum (-1)^{k+1} \alpha_{(w_1, z_1)\ldots(w_k, z_k)}(w_1, z_1) \otimes \ldots \otimes (w_k, z_k) \otimes (y, y)\\ &+ \sum (-1)^j \alpha_{(r_1, s_1)\ldots(r_k, s_k)}(r_1, s_1) \otimes \ldots \otimes (r_j, z) \otimes (z, s_j) \otimes \ldots \otimes (r_k, s_k),\end{aligned}$$

where the first two sums are over all k-tuples of basis elements of C as indicated by the subscripts of α and the third sum is a triple sum over all k-tuples of basis elements of C as indicated by the subscript of α, over $j = 1, 2, ..., k$, and for each such k-tuple and each such j over all $z \in P$ with $r_j \leqslant z \leqslant s_j$. Henceforth, the indexing of expressions derived from (5.2) will be suppressed.

Lemma 5.3. *For $k \geqslant 2$ and any $f \in Z^k(M(x, y), C)$, there exists $g \in N^{k-1}(M(x, y), C)$ such that*

$$(f - \delta g)(x, y) = \sum{}' \alpha_{(x, x_1)(x_1, x_2)...(x_{k-1}, y)} (x, x_1) \otimes (x_1, x_2) \otimes ... \otimes (x_{k-1}, y),$$

where the sum is over all $x_1, x_2, ..., x_{k-1}$ in P such that

$$x < x_1 < x_2 < ... < x_{k-1} < y.$$

Proof. As above, let

$$f(x, y) = \sum \alpha_{(u_1, v_1)...(u_k, v_k)} (u_1, v_1) \otimes ... \otimes (u_k, v_k).$$

Let $\lambda(w_1, z_1) ... (w_{k+1}, z_{k+1})$ denote the coefficient of the basis element $(w_1, z_1) \otimes ... \otimes (w_{k+1}, z_{k+1})$ in $\delta f(x, y)$. Assume that $f \in Z^k(M(x, y), C)$. Define

$$g(x, y) = \sum \beta_{(u_1, v_1)...(u_{k-1}, v_{k-1})} (u_1, v_1) \otimes ... \otimes (u_{k-1}, v_{k-1}) \in N^{k-1}(M(x, y), C)$$

as follows. Let $v_0 = x$ and $u_k = y$. Set

$$\beta_{(u_1, v_1)...(u_{k-1}, v_{k-1})} = (-1)^i \alpha_{(u_1, v_1)...(u_i, v_i)(v_i, v_i)(u_{i+1}, v_{i+1})...(u_{k-1}, v_{k-1})}$$

provided $v_j \neq u_{j+1}$ for some $j = 0, 1, ..., k-1$ in which case i is minimal among such j. Set

$$\beta_{(u_1, v_1)...(u_{k-1}, v_{k-1})} =$$

$$= (-1)^i \alpha_{(x_0, x_1)(x_1, x_2)...(x_i, x_{i+1})(x_{i+1}, x_{i+1})(x_{i+1}, x_{i+2})...(x_{k-2}, x_{k-1})}$$

provided $v_j = u_{j+1}$ for each $j = 0, 1, ..., k-1$ and provided $v_j = v_{j+1}$ for some $j = 0, 1, ..., k-2$ in which case i is minimal among such j and $x_j = v_j = u_{j+1}$ for each $j = 0, 1, 2, ..., k-1$. Otherwise set $\beta_{(u_1, v_1)...(u_{k-1}, v_{k-1})} = 0$. Let $\gamma_{(u_1, v_1)...(u_k, v_k)}$ denote the coefficient of the basis element $(u_1, v_1) \otimes ... \otimes (u_k, v_k)$ in

$\delta g(x, y) =$

$$= \sum \beta_{(u_1, v_1)\dots(u_{k-1}, v_{k-1})}(x, x) \otimes (u_1, v_1) \otimes \dots \otimes (u_{k-1}, v_{k-1})$$

$$+ \sum (-1)^k \beta_{(w_1, z_1)\dots(w_{k-1}, z_{k-1})}(w_1, z_1) \otimes \dots \otimes (w_{k-1}, z_{k-1}) \otimes (y, y)$$

$$+ \sum (-1)^j \beta_{(r_1, s_1)\dots(r_{k-1}, s_{k-1})}(r_1, s_1) \otimes \dots \otimes (r_j, z) \otimes (z, s_j) \otimes \dots \otimes (r_{k-1}, s_{k-1}).$$

$\gamma_{(u_1, v_1)\dots(u_k, v_k)}$ will be determined in a case by case manner.

First, assume that $v_j \neq u_{j+1}$ for some $j = 0, \dots, k$, where $v_0 = x$ and $u_{k+1} = y$, and let i be the minimal such j. If $i \geqslant 1$ and there exists j such that $v_j = u_{j+1}$ and $1 \leqslant j \leqslant k-1$, then define

$$\sigma_i = \sum (-1)^{j+i-1} \alpha_{(u_1, v_1)\dots(u_{j-1}, v_{j-1})(u_j, v_{j+1})(u_{j+2}, v_{j+2})\dots(u_i, v_i)(v_i, v_i)(u_{i+1}, v_{i+1})\dots(u_k, v_k)}$$

$$+ \sum (-1)^{j+i} \alpha_{(u_1, v_1)\dots(u_i, v_i)(v_i, v_i)(u_{i+1}, v_{i+1})\dots(u_{j-1}, v_{j-1})(u_j, v_{j+1})(u_{j+2}, v_{j+2})\dots(u_k, v_k)},$$

where the first sum is vacuous if $i = 1$ and otherwise is over $j = 1, 2, \dots, i-1$, and the second sum is over those j for which $i < j \leqslant k-1$ and $v_j = u_{j+1}$. Cases 3–6 below will involve σ_i.

Case 1: $i = 0$ and $(u_k, v_k) \neq (y, y)$. If $v_j \neq u_{j+1}$ for each $j = 1, 2, \dots, k-1$, then $\gamma_{(u_1, v_1)\dots(u_k, v_k)} = 0$, and to see that $\alpha_{(u_1, v_1)\dots(u_k, v_k)} = 0$ set $\lambda_{(x,x)(u_1, v_1)\dots(u_k, v_k)} = 0$. Otherwise

$$\gamma(u_1, v_1) \dots (u_k, v_k) =$$

$$= \sum (-1)^j \beta_{(u_1, v_1)\dots(u_{j-1}, v_{j-1})(u_j, v_{j+1})(u_{j+2}, v_{j+2})\dots(u_k, v_k)}$$

$$= \sum (-1)^j \alpha_{(x,x)(u_1, v_1)\dots(u_{j-1}, v_{j-1})(u_j, v_{j+1})(u_{j+2}, v_{j+2})\dots(u_k, v_k)}$$

$$= \alpha_{(u_1, v_1)\dots(u_k, v_k)},$$

where the sum is over those j for which $v_j = u_{j+1}$ and $1 \leqslant j \leqslant k-1$, and the last equality is derived from $\lambda_{(x,x)(u_1, v_1)\dots(u_k, v_k)} = 0$.

Case 2: $i = 0$ and $(u_k, v_k) = (y, y)$. Then

$$\gamma_{(u_1, v_1)\dots(u_k, v_k)} =$$

$$= \gamma_{(u_1, v_1)\dots(u_{k-1}, v_{k-1})(y, y)}$$

$$= (-1)^k \beta_{(u_1,v_1)\dots(u_{k-1},v_{k-1})}$$
$$+ \sum (-1)^j \beta_{(u_1,v_1)\dots(u_{j-1},v_{j-1})(u_j,v_{j+1})(u_{j+2},v_{j+2})\dots(u_k,v_k)}$$
$$= (-1)^k \alpha_{(x,x)(u_1,v_1)\dots(u_{k-1},v_{k-1})}$$
$$+ \sum (-1)^j \alpha_{(x,x)(u_1,v_1)\dots(u_{j-1},v_{j-1})(u_j,v_{j+1})(u_{j+2},v_{j+2})\dots(u_k,v_k)}$$
$$= \alpha_{(u_1,v_1)\dots(u_{k-1},v_{k-1})(y,y)},$$

where the sum is over those j (if any) for which $v_j = u_{j+1}$ and $1 \leqslant j \leqslant k-1$, and the last equality follows from $\lambda_{(x,x)(u_1,v_1)\dots(u_k,v_k)} = 0$.

Case 3: $i > 0$, $v_1 \neq x$ and $(u_k, v_k) \neq (y, y)$. If $i = 1$ and $v_j \neq u_{j+1}$ for each $j = 1, 2, \dots, k-1$, then

$$\gamma_{(u_1,v_1)\dots(u_k,v_k)} = \gamma_{(x,v_1)(u_2,v_2)\dots(u_k,v_k)} = 0,$$

and $\alpha_{(x,v_1)(u_2,v_2)\dots(u_k,v_k)} = 0$ since

$$\lambda_{(x,v_1)(u_2,v_2)\dots(u_{k-1},v_{k-1})(u_k,u_k)(u_k,v_k)} = 0.$$

Otherwise,

$$\gamma_{(u_1,v_1)\dots(u_k,v_k)} = \sum (-1)^j \beta_{(u_1,v_1)\dots(u_{j-1},v_{j-1})(u_j,v_{j+1})(u_{j+2},v_{j+2})\dots(u_k,v_k)}$$
$$= \sigma_i = \alpha_{(u_1,v_1)\dots(u_k,v_k)},$$

where the sum in the β's is over those j for which $v_j = u_{j+1}$ and $1 \leqslant j \leqslant k-1$. The last equality is derived from

$$\lambda_{(x,v_1)(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_k,v_k)} = 0$$

remembering that $u_1 = x$.

Case 4: $i > 0$, $v_1 \neq x$ and $(u_k, v_k) = (y, y)$. Then

$$\gamma_{(u_1,v_1)\dots(u_k,v_k)} = \gamma_{(x,v_1)(u_2,v_2)\dots(u_{k-1},v_{k-1})(y,y)}$$
$$= (-1)^k \beta_{(x,v_1)(u_2,v_2)\dots(u_{k-1},v_{k-1})}$$

$$+ \sum (-1)^j \beta_{(u_1,v_1)\dots(u_{j-1},v_{j-1})(u_j,v_{j+1})(u_{j+2},v_{j+2})\dots(u_k,v_k)}$$

$$= (-1)^{k+i} \alpha_{(x,v_1)(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_{k-1},v_{k-1})} + \sigma_i$$

$$= \alpha_{(x,v_1)(u_2,v_2)\dots(u_{k-1},v_{k-1})(y,y)},$$

where the sum in the β's is over those j for which $v_j = u_{j+1}$ and $1 \leq j \leq k-1$. The last equality follows from

$$\lambda_{(x,v_1)(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_{k-1},v_{k-1})(y,y)} = 0 .$$

Case 5: $i > 0$, $v_1 = x$ and $(u_k, v_k) \neq (y, y)$. Then

$$\gamma_{(u_1,v_1)\dots(u_k,v_k)} = \gamma_{(x,x)(u_2,v_2)\dots(u_k,v_k)}$$

$$= \beta_{(u_2,v_2)\dots(u_k,v_k)}$$

$$+ \sum (-1)^j \beta_{(u_1,v_1)\dots(u_{j-1},v_{j-1})(u_j,v_{j+1})(u_{j+2},v_{j+2})\dots(u_k,v_k)}$$

$$= (-1)^{i-1} \alpha_{(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_k,v_k)} + \sigma_i$$

$$= \alpha_{(x,x)(u_2,v_2)\dots(u_k,v_k)},$$

where the sum in the β's is over those j for which $v_j = u_{j+1}$ and $1 \leq j \leq k-1$. The last equality is derived from

$$\lambda_{(x,x)(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_k,v_k)} = 0 .$$

Case 6: $i > 0$, $v_1 = x$ and $(u_k, v_k) = (y, y)$. Then

$$\gamma_{(u_1,v_1)\dots(u_k,v_k)} = \gamma_{(x,x)(u_2,v_2)\dots(u_{k-1},v_{k-1})(y,y)}$$

$$= \beta_{(u_2,v_2)\dots(u_{k-1},v_{k-1})(y,y)} + (-1)^k \beta_{(x,x)(u_2,v_2)\dots(u_{k-1},v_{k-1})}$$

$$+ \sum (-1)^j \beta_{(u_1,v_1)\dots(u_{j-1},v_{j-1})(u_j,v_{j+1})(u_{j+2},v_{j+2})\dots(u_k,v_k)}$$

$$= (-1)^{i-1} \alpha_{(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_{k-1},v_{k-1})(y,y)}$$

$$+(-1)^{k+i}\alpha_{(u_1,v_1)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_{k-1},v_{k-1})}+\sigma_i$$

$$=\alpha_{(x,x)(u_2,v_2)\dots(u_{k-1},v_{k-1})(y,y)}\,,$$

where the sum in the β's is over those j for which $1 \leqslant j \leqslant k-1$ and $v_j = u_{j+1}$. The last equality is derived from

$$\lambda_{(x,x)(u_2,v_2)\dots(u_i,v_i)(v_i,v_i)(u_{i+1},v_{i+1})\dots(u_{k-1},v_{k-1})(y,y)}=0\,.$$

This completes the case where $v_j \neq u_{j+1}$ for some j with $0 \leqslant j \leqslant k$. Now assume that $u_1 = x$, $v_k = y$, and that $v_j = u_{j+1}$ for each $j = 1, 2, \dots, k-1$. Let $x_0 = x$, $x_k = y$, and $x_j = v_j$ for $j = 1, 2, \dots, k-1$. So

$$x = x_0 \leqslant x_1 \leqslant x_2 \leqslant \dots \leqslant x_{k-1} \leqslant x_k = y\,.$$

Further assume that $x_j = x_{j+1}$ for some j with $0 \leqslant j \leqslant k-1$, and let i be minimal among such j. $\gamma_{(x_0,x_1)\dots(x_{k-1},x_k)}$ will be computed in a case by case procedure.

Case 7: $i = 0$ and $x_{k-1} = y$. Then

$$\gamma_{(x_0,x_1)\dots(x_{k-1},x_k)}=$$

$$=\gamma_{(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)}$$

$$=\beta_{(x_0,x_2)(x_2,x_3)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)}$$

$$+(-1)^k\beta_{(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)}$$

$$+\sum_{j=1,\dots,k-1}(-1)^j\beta_{(x_0,x_1)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$=\sum_{j=2,\dots,k-2}(-1)^j\beta_{(x_0,x_1)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$=\sum_{j=2,\dots,k-2}(-1)^j\alpha_{(x_0,x_1)(x_1,x_1)(x_1,x_2)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$=\alpha_{(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)}\,,$$

where the last equality follows from

$$\lambda_{(x_0,x_0)(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)}=0$$

remembering that $x_0=x_1$ and $x_{k-1}=x_k$.

Case 8: $i=0$ and $x_{k-1}\neq y$. Then

$$\gamma_{(x_0,x_1)\dots(x_{k-1},x_k)}=$$

$$=\gamma_{(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-1},x_k)}$$

$$=\beta_{(x_0,x_2)(x_2,x_3)\dots(x_{k-1},x_k)}$$

$$+\sum_{j=1,\dots,k-1}(-1)^j\beta_{(x_0,x_1)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$=\sum_{j=2,\dots,k-1}(-1)^j\beta_{(x_0,x_1)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$=\sum_{j=2,\dots,k-1}(-1)^j\alpha_{(x_0,x_1)(x_1,x_1)(x_1,x_2)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$=\alpha_{(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-1},x_k)},$$

where the last equality follows from

$$\lambda_{(x_0,x_0)(x_0,x_0)(x_0,x_2)(x_2,x_3)\dots(x_{k-1},x_k)}=0$$

recalling that $x_0=x_1$.

Case 9: $i>0$ and $x_{k-1}=y$. Then

$$\gamma_{(x_0,x_1)\dots(x_{k-1},x_k)}=$$

$$=\gamma_{(x_0,x_1)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)}$$

$$=(-1)^k\beta_{(x_0,x_1)\dots(x_{k-3},x_{k-2})(x_{k-2},x_k)}$$

$$+\sum_{j=1,\dots,k-1}(-1)^j\beta_{(x_0,x_1)\dots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\dots(x_{k-1},x_k)}$$

$$= \sum_{j=1,\ldots,i-1,i+2,\ldots,k-2} (-1)^j \beta_{(x_0,x_1)\ldots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{k-1},x_k)}$$

$$= \sum_{j=1,\ldots,i-1} (-1)^{j+i-1} \alpha_{(x_0,x_1)\ldots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{i-1},x_i)(x_i,x_{i+1})(x_{i+1},x_{i+1})}$$
$$_{(x_{i+1},x_{i+2})\ldots(x_{k-1},x_k)}$$

$$+ \sum_{j=i+2,\ldots,k-2} (-1)^{j+i} \alpha_{(x_0,x_1)\ldots(x_{i-1},x_i)(x_i,x_{i+1})(x_{i+1},x_{i+1})(x_{i+1},x_{i+2})\ldots}$$
$$_{(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{k-1},x_k)}$$

$$= \alpha_{(x_0,x_1)\ldots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)},$$

where the last equality is derived from

$$\lambda_{(x_0,x_1)\ldots(x_{i-1},x_i)(x_i,x_{i+1})(x_{i+1},x_{i+1})(x_{i+1},x_{i+2})\ldots(x_{k-3},x_{k-2})(x_{k-2},x_k)(x_k,x_k)} = 0$$

remembering that $x_{k-1} = x_k$. This computation holds even when $i = k-1$ provided the second sum in the α's is interpreted to be vacuous.

Case 10: $i > 0$ and $x_{k-1} \neq y$. Then

$$\gamma_{(x_0,x_1)\ldots(x_{k-1},x_k)} =$$

$$= \sum_{j=1,\ldots,k-1} (-1)^j \beta_{(x_0,x_1)\ldots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{k-1},x_k)}$$

$$= \sum_{j=1,\ldots,i-1,i+2,\ldots,k-1} (-1)^j \beta_{(x_0,x_1)\ldots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{k-1},x_k)}$$

$$= \sum_{j=1,\ldots,i-1} (-1)^{j+i-1} \alpha_{(x_0,x_1)\ldots(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{i-1},x_i)}$$
$$_{(x_i,x_{i+1})(x_{i+1},x_{i+1})(x_{i+1},x_{i+2})\ldots(x_{k-1},x_k)}$$

$$+ \sum_{j=i+2,\ldots,k-1} (-1)^{j+i} \alpha_{(x_0,x_1)\ldots(x_{i-1},x_i)(x_i,x_{i+1})(x_{i+1},x_{i+1})(x_{i+1},x_{i+2})\ldots}$$
$$_{(x_{j-2},x_{j-1})(x_{j-1},x_{j+1})(x_{j+1},x_{j+2})\ldots(x_{k-1},x_k)}$$

$$= \alpha_{(x_0,x_1)\ldots(x_{k-1},x_k)},$$

where the last equality follows from

$$\lambda_{(x_0,x_1)\dots(x_{i-1},x_i)(x_i,x_{i+1})(x_{i+1},x_{i+1})(x_{i+1},x_{i+2})\dots(x_{k-1},x_k)} = 0 .$$

This finally establishes that

$$\gamma_{(u_1,v_1)\dots(u_k,v_k)} = \alpha_{(u_1,v_1)\dots(u_k,v_k)}$$

except in the case where $u_1 = x$, $v_k = y$, $v_j = u_{j+1}$ for $j = 1, 2, \dots, k-1$, and $u_j < v_j$ for $j = 1, 2, \dots, k$. But in this case, letting $x_0 = x$, $x_k = y$, and $x_j = v_j$ for $j = 1, 2, \dots, k-1$ so that $x = x_0 < x_1 < x_2 \dots < x_{k-1} < x_k = y$, it follows easily that $\gamma_{(x_0,x_1)\dots(x_{k-1},x_k)} = 0$, completing the proof of the lemma.

Theorem 5.4. *If for every proper chain*

$$x = x_0 < x_1 < x_2 < \dots < x_{s-1} < x_s = y$$

from x to y, $s < k$, then $H^k(M(x,y), C) = 0$.

Proof. This is an immediate consequence of Theorems 5.1 and 5.2 and Lemma 5.3.

6. Cohomology of a geometric lattice

Let $P = G$ be a geometric lattice with rank function r (see [1] for definitions). The preceding results, as they apply to his case, will be interpreted, and more computations, dependent on this additional structure, will be performed.

Recall that the Whitney number $w_2(k)$ of the second kind is just the number of elements of G of rank k and that the Whitney number $w_1(k)$ of the first kind is defined by

$$w_1(k) = \sum_{\substack{x\in G\\ r(x)=k}} \mu(0,x) ,$$

where μ is the Möbius function of G.

Let $M = \bigoplus_{x\in G} M(0,x)$. Theorem 5.4 becomes

Theorem 6.1. *$H^k(M(x,y), C) = (0)$ for $k > r(y) - r(x)$.*

Thus, it follows that

Theorem 6.2. $H^k(M, C) = (0)$ *for all* $k > r(G)$.

Theorem 6.3. $\dim H^0(M, C) = 1 = w_1(0) = w_2(0)$.

Proof. This follows directly from Theorem 5.1.

Theorem 5.2 gives the following two theorems:

Theorem 6.4. $\dim H^1(M(x, y), C) = 1$ *or* 0 *depending respectively on whether* $r(y) - r(x) = 1$ *or not.*

Theorem 6.5. $\dim H^1(M, C) = w_2(1) = -w_1(1)$.

Theorem 6.6. $H^2(M(x, y), C) = 0$ *if* $r(y) - r(x) > 2$.

Proof. Let $f \in Z^2(M(x, y), C)$. By Lemma 5.3, it may be assumed that

$$f(x, y) = \sum_{x < t < y} \alpha_{(x, t)(t, y)}(x, t) \otimes (t, y) .$$

It will be shown that $\alpha_{(x, t)(t, y)} = \alpha_{(x, s)(s, y)}$ for any $x < t < y$ and $x < s < y$. Suppose first that for such s and t with $t \neq s$ that $r(s) - r(x) = 1 = r(t) - r(x)$. Let $u = s \vee t$. Since G is a geometric lattice, $r(u) - r(x) = 2$. Since $r(y) - r(x) > 2$, $x < s, t < u < y$. But $f \in Z^2(M(x, y), C)$ so that the coefficients of $(x, t) \otimes (t, u) \otimes (u, y)$ and $(x, s) \otimes (s, u) \otimes (u, y)$ in (5.2) may be set to 0 to yield $\alpha_{(x, t)(t, y)} = \alpha_{(x, u)(u, y)} = \alpha_{(x, s)(s, y)}$. Hence $\alpha_{(x, t)(t, y)} = \alpha_{(x, s)(s, y)}$ for all s and t such that $x < s, t < y$ and $r(s) - r(x) = 1 = r(t) - r(x)$. Now for any z such that $x < z < y$ there exists t such that $x < t \leqslant z$ and $r(t) - r(x) = 1$. So setting the coefficient of $(x, t) \otimes (t, z) \otimes (z, y)$ in (5.2) to 0 gives $\alpha_{(x, t)(t, y)} = \alpha_{(x, z)(z, y)}$. Hence, $\alpha_{(x, t)(t, y)} = \alpha_{(x, s)(s, y)}$ for any s, t with $x < s, t < y$. Finally, define $g \in N^1(M(x, y), C)$ by $g(x, y) = \alpha_{(x, t)(t, y)}(x, y)$, where t is any element of G such that $x < t < y$. Then it is easily checked that $\delta g = f$.

Theorem 6.7. $\dim H^2(M(x, y), C) = \mu(x, y)$ *when* $r(y) - r(x) = 2$.

Proof. By Lemma 5.3, it may be assumed that $f \in Z^2(M(x, y), C)$ has the form

$$f(x, y) = \sum_{x<t<y} \alpha_{(x, t)(t, y)}(x, t) \otimes (t, y) .$$

Setting (5.2) to 0 ($\delta f = 0$) gives no relations among the $\alpha_{(x, t)(t, y)}$'s, and so dim $Z^2(M(x, y), C)$ is just the number of t with $x < t < y$. But this number is just $\mu(x, y) + 1$ since for $x < t < y$, $r(y) - r(t) = 1 = r(t) - r(x)$ and $\mu(x, t) = -1$. For any $g \in N^1(M(x, y), C)$ and for any $x < t, s < y$, the coefficients of $(x, t) \otimes (t, y)$ and $(x, s) \otimes (s, y)$ in $\delta g(x, y)$ are equal and may be set to be non-zero. It follows that dim $H^2(M(x, y), C) = \mu(x, y)$.

Theorem 6.8. dim $H^2(M, C) = w_1(2)$.

Let β_k = dim $H^k(M, C)$. It has now been proved that $\beta_k = (-1)^k w_1(k)$ for $k = 0, 1, 2$, and that $\beta_k = 0$ for $k > r(G)$. It would of course be interesting if $\beta_k = (-1)^k w_1(k)$ for $k = 3, 4, \ldots, r(G)$ could be established.

References

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